Some new observations on the t-equivalence relation

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Terminology and notation

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Let $\mathcal{P}$ be a topological property. Suppose, that X has $\mathcal{P}$ and that X and Y are t-equivalent. Does Y has property $\mathcal{P}$?

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Examples

1. If X and Y are t-equivalent and X is σ-compact, then Y is σ-compact
2. $[0,1]$ and $\mathbb{R}$ are t-equivalent so compactness is not preserved by t-equivalence.

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Suppose X and Y are t-equivalent. Is it true that $\dim X = \dim Y$?

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The answer is affirmative under a stronger assumption: $C_p(X)$ and $C_p(Y)$ are **linearly** or **uniformly** homeomorphic.
In particular, we do not know if the spaces $C_p(2^\omega)$ and $C_p([0,1])$ or the spaces $C_p([0,1])$ and $C_p([0,1]^2)$ are homeomorphic.

Three notions of infinite-dimension

A normal space X is called a C-space if for any sequence of its open covers $(\mathcal{U}_i)_{i\in\omega}$, there exists a sequence $(\mathcal{V}_i)_{i\in\omega}$ of families of pairwise disjoint open sets such that $\mathcal{V}_i$ is a refinement of $\mathcal{U}_i$ and $\bigcup_{i\in\omega}\mathcal{V}_i$ is a cover of X.

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A normal space X is called weakly infinite-dimensional if for every sequence $(A_1, B_1), (A_2, B_2), \ldots$ of pairs of disjoint closed subsets of X there exists closed sets $L_1, L_2, \ldots$ such that L_i is a partition between A_i and B_i and $\bigcap_i L_i = \emptyset$.

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Fact

countable-dimensional metrizable $\subseteq$ C-space $\subseteq$ weakly infinite-dimensional

Theorem (Cauty 1999)

Let X and Y be metrizable compact spaces such that $C_p(Y)$ is an image of $C_p(X)$ under a continuous open mapping. If for all $n \in \mathbb{N}^+$ the space X^n is weakly infinite-dimensional, then for all $n \in \mathbb{N}^+$ the finite power Y^n is also weakly infinite-dimensional.

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Theorem (K.)

Let X and Y be σ-compact metrizable spaces. Suppose, that $C_p(Y)$ is an image of $C_p(X)$ under a continuous open mapping. If X is a C-space, then Y is also a C-space.

On a theorem of Okunev

Theorem (Okunev 2011)

Suppose that there is an open continuous surjection from $C_p(X)$ onto $C_p(Y)$. Then there are spaces Z_n , locally closed subspaces B_n of Z_n, and locally closed subspaces Y_n of Y, $n \in \mathbb{N}^+$, such that each Z_n admits a perfect finite-to-one mapping onto a closed subspace of X^n, Y_n is an image under a perfect mapping of B_n, and $Y = \bigcup\{Y_n : n \in \mathbb{N}^+\}$.

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- The set Z_n is σ-compact (as a perfect preimage of a σ-compact set) Since for compact spaces the property C is preserved by continuous mappings with fibers of cardinality $< \mathfrak{c}$ we conclude that Y_n and thus Y is a C-space.

Further results

Theorem (Cauty 1999)

Let X and Y be metrizable compact spaces such that $C_p(Y)$ is an image of $C_p(X)$ under a continuous open mapping. If for all $n \in \mathbb{N}^+$ the space X^n is weakly infinite-dimensional, then for all $n \in \mathbb{N}^+$ the finite power Y^n is also weakly infinite-dimensional.

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A normal space X is called a *k-C-space*, where $k \geqslant 2$, if for any sequence of its covers $(\mathcal{U}_i)_{i\in\omega}$, $|\mathcal{U}_i| \leqslant k$, there exists a sequence $(\mathcal{V}_i)_{i\in\omega}$ of families of pairwise disjoint open sets such that for every $i \in \omega$ the family $\mathcal{V}_i$ refines $\mathcal{U}_i$ and $\bigcup_{i\in\omega} \mathcal{V}_i$ covers X.

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A space is weakly infinite-dimensional iff it is a 2-C-space

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Theorem (K.)

Let X and Y be metrizable σ-compact spaces such that $C_p(Y)$ is an image of $C_p(X)$ under a continuous open mapping. Fix $k \geqslant 2$. If for all $n \in \mathbb{N}^+$ the space X^n is a k-C-space, then Y is also a k-C-space.

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Question

Can we get rid of the metrizability assumption in the above theorems?